Period 3, Oct 7, 2024

66. $y = \cos\sqrt{\sin(\tan \pi x)}$ $\Rightarrow y = \cos\sqrt{\sin(\tan u)}$ $\Rightarrow y = \cos\sqrt{\sin L}$ $\Rightarrow y = \cos\sqrt{m}$ $\Rightarrow y = \cos p$

$u = \pi x$ $\qquad L = \tan u$ $\qquad m = \sin L$ $\qquad p = \sqrt{m} = m^{\frac{1}{2}}$ $\qquad \frac{dy}{dp} = -\sin p$

$\frac{du}{dx} = \pi$ $\qquad \frac{dL}{du} = \sec^2 u$ $\qquad \frac{dm}{dL} = \cos L$ $\qquad \frac{dp}{dm} = \frac{1}{2\sqrt{m}} = \frac{1}{2m^{\frac{1}{2}}} = \frac{1}{2}m^{-\frac{1}{2}}$

$$\frac{\cancel{du}}{dx}\cdot\frac{\cancel{dL}}{\cancel{du}}\cdot\frac{\cancel{dm}}{\cancel{dL}}\cdot\frac{dp}{\cancel{dm}}\cdot\frac{dy}{dp}$$

$$\pi \cdot \sec^2 \pi x \cos(\tan \pi x)\frac{1}{2\sqrt{\sin(\tan \pi x}} \cdot -\sin\sqrt{\sin(\tan(\pi x))}$$

---

52. $g(\theta) = \sec(\frac{1}{2}\theta)\tan(\frac{1}{2}\theta)$ $\Rightarrow g(u) = \underbrace{\sec u}_{F(u)}\underbrace{\tan u}_{h(u)}$

$u = \frac{1}{2}\theta$

$\frac{du}{d\theta} = \frac{1}{2}$ $\qquad g'(u) = F'(u)h(u) + F(u)\cdot h'(u)$

$$\frac{dy}{du} = g'(u) = \sec u \tan u \cdot \tan u + \sec u \cdot \sec^2 u$$

$$\frac{dy}{d\theta} = \frac{1}{2}\left[\sec\tfrac{1}{2}\theta\tan^2\tfrac{1}{2}\theta + \sec^3\tfrac{1}{2}\theta\right]$$

---

51. $h(x) = \sin 2x \cos 2x$

$F(x) = \sin 2x$ $\qquad g(x) = \cos 2x$

$F'(x) = 2\cos 2x$ $\qquad g'(x) = -2\sin 2x$

$u = 2x$

$\frac{du}{dx} = 2$

$h'(x) = 2\cos 2x \cdot \cos 2x + \sin 2x(-2\sin 2x)$

$2\cos^2 2x - 2\sin^2 2x$

$2(\cos^2 2x - \sin^2 2x) = 2\cos 4x$

$$\frac{1}{2\sqrt{x}}\cdot 1\cdot\frac{1}{2\sqrt{L}}\cdot\frac{1}{2\sqrt{m}}$$

$$\frac{1}{2\sqrt{x}}\cdot\frac{1}{2\sqrt{2+\sqrt{x}}}\cdot\frac{1}{2\sqrt{2+\sqrt{2+\sqrt{x}}}}$$

35. $f(x) = \sqrt{2+\sqrt{2+\sqrt{x}}}$

$u = \sqrt{x} = x^{\frac{1}{2}}$

$\frac{du}{dx} = \frac{1}{2\sqrt{x}} = \frac{1}{2}x^{-\frac{1}{2}}$

$y = \sqrt{2+\sqrt{2+u}}$

$L = 2+u$

$\frac{dL}{du} = 1$

$y = \sqrt{2+\sqrt{L}}$

$m = 2+\sqrt{L} = 2+L^{\frac{1}{2}}$

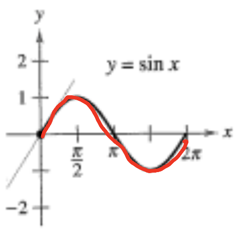
$\frac{dm}{dL} = 0 + \frac{1}{2}L^{\frac{1}{2}-1=-\frac{1}{2}} = \frac{1}{2\sqrt{L}}$

$y = \sqrt{m} = m^{\frac{1}{2}}$

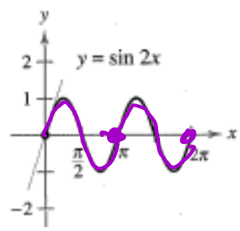
$\frac{dy}{dm} = \frac{1}{2}m^{-\frac{1}{2}} = \frac{1}{2\sqrt{m}}$

In Exercises 43 and 44, find the slope of the tangent line to the sine function at the origin. Compare this value with the number of complete cycles in the interval $[0, 2\pi]$. What can you conclude about the slope of the sine function $\sin ax$ at the origin?

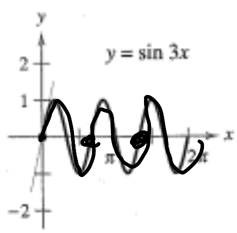
43. (a)



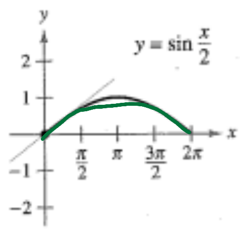
(b)



44. (a)



(b)



43

$y = \sin x$

$\frac{dy}{dx} = \cos x$ 1 period

$y = \sin 2x$ 2 periods

$u = 2x$

$\frac{dy}{dx} = 2\cos 2x$

44

$y = \sin 3x$ 3 periods

$u = 3x$

$\frac{dy}{dx} = 3\cos 3x$

$y = \sin \frac{x}{2}$ $\frac{1}{2}$ period

$\frac{dy}{dx} = \frac{1}{2}\cos\frac{x}{2}$

$y = \sin\frac{x}{2} \Rightarrow y = \sin u$

$u = \frac{x}{2} = \frac{1}{2}x$ $\frac{dy}{du} = \cos u$

$\frac{du}{dx} = \frac{1}{2}$ $\frac{du}{dx} \cdot \frac{dy}{du} = \frac{1}{2}(\cos u) = \frac{1}{2}(\cos\frac{x}{2})$

---

62. $y = 3x - 5\cos(\pi x)^2$

64. $y = \sin \sqrt[3]{x} + \sqrt[3]{\sin x}$

$\cos^2 \pi x = (\cos \pi x)^2$

$\cos(\pi x)^2 = \cos(\pi^2 x^2)$

$y = 3x - 5\cos(\pi x)^2$

$\frac{dy}{dx} = 3 - 5 \cdot \frac{d}{dx}\left(\cos(\pi x)^2\right) = \boxed{3 - 5 \cdot 2\pi^2 x \cdot -\sin(\pi x)^2}$

$y = \cos(\pi x)^2$

$u = \pi x$

$\frac{du}{dx} = \pi$

$\cos \pi^2 x^2 \Rightarrow y = \cos u$

$u = \pi^2 x^2$ $\frac{dy}{du} = -\sin u$

$\frac{du}{dx} = 2\pi^2 x$

$$y = \sin\sqrt[3]{x} + \sqrt[3]{\sin x} \Rightarrow \frac{dy}{dx} = F'(x) + g'(x)$$

$F(x) = \sin\sqrt[3]{x}$

$u = \sqrt[3]{x} = x^{\frac{1}{3}}$

$\frac{du}{dx} = \frac{1}{3} \cdot x^{-\frac{2}{3}} = \frac{1}{3\sqrt[3]{x^2}}$

$F(u) = \sin u$

$F'(u) = \cos u = \frac{dy}{du}$

$\frac{1}{3\sqrt[3]{x^2}} \cdot \cos(\sqrt[3]{x})$

$g(x) = \sqrt[3]{\sin x}$

$u = \sin x$

$\frac{du}{dx} = \cos x$

$g(u) = \sqrt[3]{u} = u^{\frac{1}{3}}$

$g'(u) = \frac{dy}{du} = \frac{1}{3\sqrt[3]{u^2}}$

$\frac{1}{3\sqrt[3]{\sin^2 x}} \cdot \cos x$

$$\frac{dy}{dx} = \frac{1}{3\sqrt[3]{x^2}} \cdot \cos\sqrt[3]{x} + \cos x \cdot \frac{1}{3\sqrt[3]{\sin^2 x}}$$

55. $y = 4\sec^2 x \Rightarrow y = 4u^2$

$u = \sec x$

$\frac{du}{dx} = \sec x \tan x$

$\frac{dy}{du} = 8u$

$\frac{\cancel{du}}{dx} \cdot \frac{dy}{\cancel{du}} = \sec x \tan x \cdot 8 \sec x$

$8\sec^2 x \tan x$

27. $y = \frac{x}{\sqrt{x^2+1}}$

$$\frac{dy}{dx} = \frac{1(\sqrt{x^2+1}) - x \cdot \frac{d}{dx}(\sqrt{x^2+1})}{(\sqrt{x^2+1})^2}$$

$y = \sqrt{x^2+1} \Rightarrow y = \sqrt{u} = u^{\frac{1}{2}}$

$u = x^2+1$

$\frac{du}{dx} = 2x$

$\frac{dy}{du} = \frac{1}{2u^{\frac{1}{2}}} = \frac{1}{2\sqrt{u}}$

$\frac{du}{dx} \cdot \frac{dy}{du} = \cancel{2}x \cdot \frac{1}{\cancel{2}\sqrt{u}} = \frac{x}{\sqrt{x^2+1}}$

$$\frac{dy}{dx} = \frac{\sqrt{x^2+1} - x\left(\frac{x}{\sqrt{x^2+1}}\right)}{(\sqrt{x^2+1})^2}$$

$$= \frac{\frac{\sqrt{x^2+1} \cdot \sqrt{x^2+1}}{\sqrt{x^2+1}} - \frac{x^2}{\sqrt{x^2+1}}}{x^2+1} = \frac{\frac{\cancel{x^2} + 1 - \cancel{x^2}}{\sqrt{x^2+1}}}{\frac{(x^2+1)}{1}} = \frac{1}{\sqrt{x^2+1}} \cdot \frac{1}{(x^2+1)}$$

$$\frac{1}{(x^2+1)^{3/2}} = \frac{1}{(x^2+1)\sqrt{x^2+1}}$$

5. The number of gallons of water in a tank t minutes after the tank has started to drain is $Q(t) = 200(30-t)^2$.

a) How fast is the water running out at the end of 10 minutes?

$$\frac{d}{dT}(Q(T)) = \frac{d}{dT}\left(200(30-T)^2\right) = 200\cdot 2(30-T)^1\cdot -1$$

$$\frac{dQ}{dT} = -400(30-T)$$

$y = 200(30-T)^2$

$u = 30-T$

$\frac{du}{dT} = -1$

$y = 200u^2$

$\frac{dy}{du} = 200\cdot 2u$

at 10 minutes

$\frac{dQ}{dT} = -400(30-10)$

$-400(20) = -8000$

b) What is the average rate at which the water flows out during the first 10 minutes?

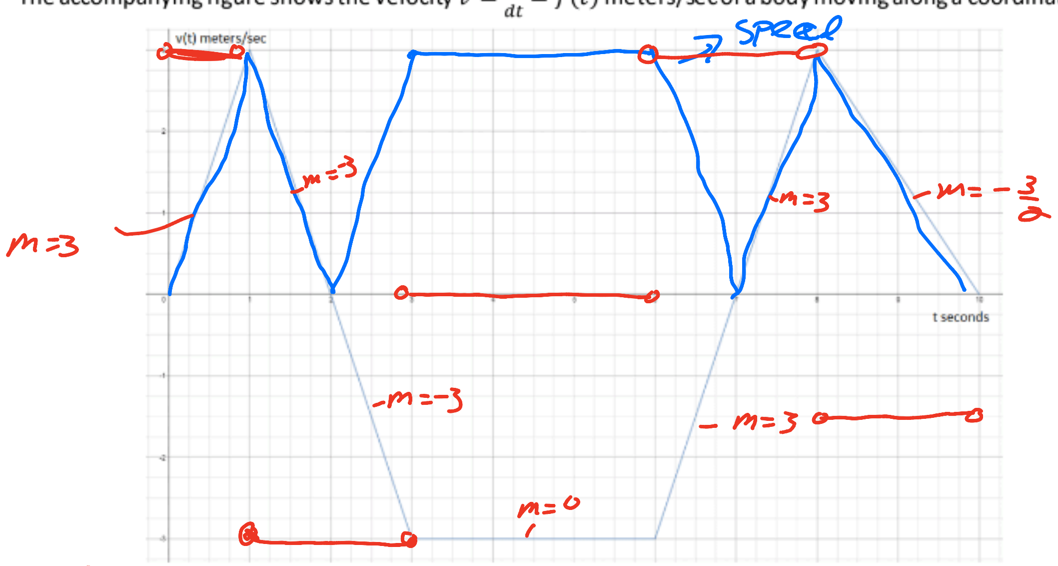
$Q(0) = 200(30-0)^2 = 200\cdot 900 = 180{,}000$

$Q(10) = 200(30-10)^2 = 200\cdot 400 = 80{,}000$

$(0, 180{,}000)$, $(10, 80{,}000)$

Find The Slope

2. The accompanying figure shows the velocity $v = \frac{ds}{dt} = f(t)$ meters/sec of a body moving along a coordinate line.



a) ~~When does the~~ body reverse direction?

Speed = |Velocity|

b) When (~~approximately~~) is the body moving at a constant speed?

c) Graph the body's speed for $0 \le t \le 10$ on the same graph in a different color.

d) Graph the acceleration, where defined, on the same grid in a different color.

acceleration = Slope of Velocity

1. A rock thrown vertically upward from the surface of the moon at a velocity of 24 m/sec (about 86 km/h) reaches a height of $s(t) = 24t - 0.8t^2$ meters in t seconds.

S(T) = height

Takes off T=0 h = 0 = S(0)

Lands T=? h = 0 = S(T)

e) How long was the rock aloft?

$0 = 24T - 0.8T^2$

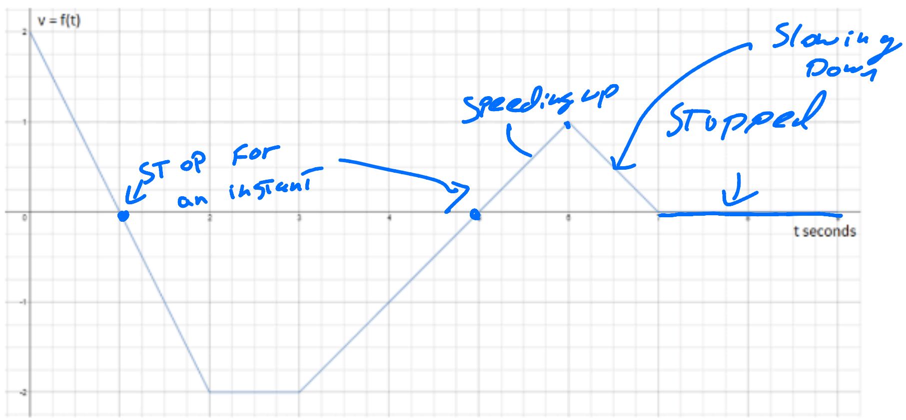
$0 = T(24 - 0.8T)$

$T = 0$ or $24 - 0.8T = 0$

Take off $24 = 0.8T$

$30 = T$ Landing

4. The accompanying figure shows the velocity $v = \frac{ds}{dt} = f(t)$ meters/sec of a body moving along a coordinate line.



a) When does the particle move forward? Move backward? Speed up? Slow down?

b) When is the particle's acceleration positive? Negative? Zero?

c) When does the particle move at its greatest speed?

d) When does the particle stand still for more than an instant?

velocity = 0

$\sqrt{y} = y^{\frac{1}{2}}$

$x^2 y^4 + 3x^6 - \sqrt{y} = \sin y$

with respect to x

$2x \cdot y^4 + x^2 \cdot 4y^3 \frac{dy}{dx} + 18x^5 - \frac{1}{2\sqrt{y}} \frac{dy}{dx} = (\cos y)\frac{dy}{dx}$

$2xy^4 + 18x^5 = \cos y \frac{dy}{dx} - 4x^2 y^3 \frac{dy}{dx} + \frac{1}{2\sqrt{y}} \frac{dy}{dx}$

$$\frac{2xy^4 + 18x^5}{\cos y - 4x^2y^3 + \frac{1}{2\sqrt{y}}} = \frac{\cancel{\frac{dy}{dx}\left(\cos y - 4x^2y^3 + \frac{1}{2\sqrt{y}}\right)}}{\cancel{\cos y - 4x^2y^3 + \frac{1}{2\sqrt{y}}}}$$

$$\frac{2xy^4 + 18x^5}{\left(\cos y - 4x^2y^3 + \frac{1}{2\sqrt{y}}\right)} = \frac{dy}{dx}$$

$y = 3x^2$

$dy = 3 \cdot 2x^1 \cdot dx$

with respect to x

$\frac{dy}{dx} = \frac{6x\cancel{dx}}{\cancel{dx}}$

$\frac{dy}{dx} = 6x$

$xy = 1$

$1 \cdot y + x \cdot \frac{dy}{dx} = 0$

$\frac{\cancel{x}\frac{dy}{dx}}{\cancel{x}} = \frac{-y}{x}$

$\frac{dy}{dx} = \frac{-y}{x}$

$\frac{d}{dx}[x^2] = 2x$ $\qquad$ $\frac{d}{dx}[x^2] = 2x$

$\frac{d}{dy}[y^2] = 2y$ $\qquad$ $\frac{d}{dx}[y^2] = 2y\frac{dy}{dx}$

$\frac{d}{dt}[t^2] = 2T$ $\qquad$ $\frac{d}{dx}[T^2] = 2T\frac{dT}{dx}$

$$\frac{d}{dt}[t^2 + x^2 + y^2] = 2T + 2x\frac{dx}{dT} + 2y\frac{dy}{dT}$$

$$\frac{d}{dx}[t^2 + x^2 + y^2] = 2T\frac{dT}{dx} + 2x + 2y\frac{dy}{dx}$$

$$\frac{d}{dy}[t^2 + x^2 + y^2] = 2T\frac{dT}{dy} + 2x\frac{dx}{dy} + 2y\cancel{\frac{dy}{dy}}$$

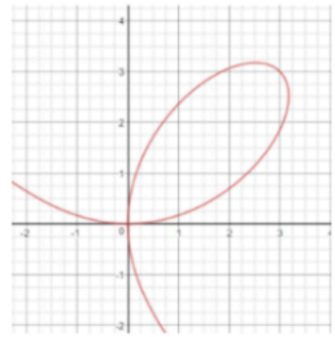
Find $\dfrac{dy}{dx}$ given that $y^3 + y^2 - 5y - x^2 = -4$

$$3y^2\frac{dy}{dx} + 2y\frac{dy}{dx} - 5\frac{dy}{dx} \cancel{-2x} = 0$$

$$\cancel{+2x} \qquad +2x$$

$$\frac{\frac{dy}{dx}\cancel{(3y^2+2y-5)}}{\cancel{3y^2+2y-5}} = \frac{2x}{3y^2+2y-5}$$

$$\frac{dy}{dx} = \frac{2x}{3y^2+2y-5}$$



Example 4a:

Find $\frac{dy}{dx}$ given that $x^3+y^3=6xy$

$$x^3+y^3=(6x)y$$

$$3x^2+3y^2\frac{dy}{dx}=6y+6x\frac{dy}{dx}$$

$$3x^2-6y=6x\frac{dy}{dx}-3y^2\frac{dy}{dx} \Rightarrow 3x^2-6y=\frac{dy}{dx}(6x-3y^2)$$

$$\frac{3x^2-6y}{6x-3y^2}=\frac{dy}{dx}$$

Line (Slope and a Point)

Example 4b: Find the equation of the tangent and normal line to the graph in 4a at the point $\left(\frac{4}{3},\frac{8}{3}\right)$.

$$y-\frac{8}{3}=\frac{4}{5}\left(x-\frac{4}{3}\right)$$

Point

$$\frac{dy}{dx}=\frac{3\left(\frac{4}{3}\right)^2-6\left(\frac{8}{3}\right)}{6\left(\frac{4}{3}\right)-3\left(\frac{8}{3}\right)^2}=\frac{\frac{16}{3}-\frac{48}{3}}{\frac{24}{3}-\frac{64}{3}}=\frac{\frac{-32}{3}}{\frac{-40}{3}}$$

$$+\frac{32}{\cancel{3}}\cdot\frac{\cancel{3}}{+40}=\frac{4\cdot\cancel{8}}{5\cdot\cancel{8}}=\frac{4}{5}=\text{Slope}$$

Normal Line

$\left(\frac{4}{3},\frac{8}{3}\right)$ $\quad m=\frac{-5}{4}$

$$y-\frac{8}{3}=\frac{-5}{4}\left(x-\frac{4}{3}\right)$$

$$y-\frac{8}{3}=-\frac{5}{4}x+\frac{20}{12}+\frac{8\cdot4}{3\cdot4}$$

$+\frac{8}{3}$

$$y=\frac{-5}{4}x+\frac{20}{12}+\frac{32}{12}\Rightarrow y=\frac{-5}{4}x+\frac{52}{12}=\frac{13}{3}$$

$$y=\frac{-5}{4}x+\frac{13}{3}$$

$$y=\frac{-5}{4}x+4\frac{1}{3}$$

Example 5: Find $\frac{dy}{dx}$ at the point $(0,0)$ of the function $\tan(x+y)=x$

$u=x+y$

$\frac{du}{dx}=1+\frac{dy}{dx}$

$\tan u=x$

$(\sec^2 u)\frac{du}{dx}=1$

$\sec^2(x+y)\left[1+\frac{dy}{dx}\right]=1$

$\sec^2(x+y)+\sec^2(x+y)\cdot\frac{dy}{dx}=1$